Use of the Helmholtz decomposition in search for equilibria in games

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Abstract

We investigate the use of the Helmholtz decomposition to approximate pure and mixed equilibria in games. The method relies on a representation of strategic games as graphs endowed with flows, where the flows represent changes in utility arising from players moving between actions. Specifically, we investigate a ranking game, a congestion game for two players, and a generic decomposition of a game for three players, each with different actions sets.

Contents

1 Introduction

Game theory is to scenarios of strategy what probability is to scenarios of chance.

In this text we aim to investigate the use of Helmholtz decomposition to find optimal solutions to games. We will use the theory to decompose flows on graphs, where the graphs under investigation are constructed from games. Games in turn are theoretical models that allow us to study scenarios where the intentional choices of participants, called *players*, to act in some specified way have an impact on the outcome of the model. The flows represent changes in benefit derived, *utility*, by players as they change their actions with respect to one another. A choice of actions is called a *strategy*. The theoretical framework for the particular decompositions studied in this essay has been developed primarily by Candogan, Menache, Ozdaglar and Parrilo [2]. *Game theory* is the name given to the general tool set developed to study situations where the choices of players in a model setting leads to better or worse outcomes for the participants.

The earliest studies of game theory concentrated on the so-called *zero-sum games*. There are scenarios in which each player in order to obtain maximal gain, must cause to the other player maximal loss. These types of games are introduced in a formal and in-depth way in [4].

General-sum games include scenarios where, in order for one player to win, the other players must not necessarily lose. These types of games are common in economics, and a comprehensive introduction to these games as well as algorithms to solve them may be found in [5] or [3].

In [3, Ch. 9.5] and [13] we get an overview of the use of game theory in *voting games* and *ranking systems*. These types of games are used to describe democratic processes (for instance, how to justly elect a most preferred representative) or preferences (for instance, whether one piece of music is over-all more preferred by a group than another piece of music). An example application from the financial industries is given below in section 4.1.

Recent developments in game theory include studies of so-called *public goods games*, where joint management of resources by a group is considered. A seminal contribution to this area may be found in [6]. Here we will consider *congestion games*, where joint use of a resource may cause deterioriation of the utility of the resource (section 4.2).

Finally, we may consider *stochastic games*. These games are generalisations of *Markov decision processes* (MDP:s) to more than one player, and find applications in industrial management (see [3, Ch. 6.2.1] and [10]). Since some of the choices in a stochastic game are made arbitrarily by an external player, *nature*, over which the regular players have no control, the details of defining strategy spaces for these games are challenging.

All of these examples of games may be considered *classes of games*. In addition to the above examples, there are also studies on infinite-player games, or games with infinite sets of actions that may be chosen by each player. We may consider *multi-stage games*, where players are faced with problem of maximising their outcome through sequential choices in different settings, or *repeated games*, where the players are faced with the same set of choices over and over again. We may have *perfect information games*, where all the players know the options available to all other players and the degree of utility derived from different players by making different choices, and *imperfect information games*, where the players do not generally know what choices other players make, or how much utility they may derive from making any particular choice.

The problem of finding an optimal solution for any finite-player game has been demonstrated by Chen and Deng to be *PPAD-complete*[12]. This means that the *complexity class of finite-player games* is such that, in spite of us knowing that each such game has a solution, we can in general find no efficient algorithms for locating these solutions. The fact that exact solutions may be difficult or time-consuming to find, motivates the study of how to find or approximate solutions. *Algorithmic game theory* has become a popular subject of study in computer science, where algorithms are sought to efficiently solve games.

Both zero-sum games and general-sum games may have deterministic solutions: a fixed point at which each player has clearly made a choice which, with respect to all the other choices the player can make, gives the best possible outcome for that player given the choices preferred by other players. We call these solutions *pure strategy equilibria.*

It may also be the case that, no matter what choices a player makes, it will always be beneficial for at least one of the co-players to change their choice. When the co-player changes its choice, the original choice by the player may no longer lead to the best outcome for that player and so the player has to adjust their choice as well. We call such cases *mixed strategy equilibria.*

2 Preliminaries on games of strategy

Definition 1 (*Normal, strategic-form game*)**.** A normal, strategic-form game is the tuple $(\mathcal{M}, E, \mathcal{U})$ where

1. $\mathcal{M} := \{1, ..., n\}$ are players, for some $n < \infty$.
2. $E := \prod_{m \in \mathcal{M}} E_m$, where E_m is a finite set whose every point represents an action that can be chosen by player $m \in \mathcal{M}\}$.
3. $\mathcal{U} := \{u_m : E \mapsto \mathbb{R},\ m \in \mathcal{M}\}$ is a collection of given functions called utility functions.

The *utility function* describes how good or bad the choice of a particular action is for a particular player. Similar to random variables, which assign (real) values to elementary events or sets of elementary events, utility functions assign (real) values to actions. Depending on the specific scenario to be modelled, utility functions may take on a range of different forms. We will add assumptions about the utility function, adding criteria on the utility functions as we develop specific models for further study.

Definition 2. A vector $\mathbf{p} = (p_m, m \in \mathcal{M}) \in E$ where p_m is the stategy of the m^{th} player will be called a strategy profile.

To highlight the dependence on m we will sometimes use the notation $(\mathbf{p}_m, \mathbf{p}_{m-})$ to indicate the situation that player m has made the choice of action $p_m \in E_m$ while all the other $n-1$ players have made the choices of action

$$p_{m-} \in \prod_{\substack{k \neq m \\ k \in \mathcal{M}}} E_k \ . \tag{1}$$

In particular, we usually write $u_m(p_m, p_{m-}) = u_m(\mathbf{p})$.

Game theory relates, generally, to finding equilibria in a specific game setting. Equilibria are outcomes from which none of the players can profitably deviate by shifting the game to a different strategy profile as defined in Def. 2. In an equilibrium, every change of action by either player will cause that player to gain a lower utility. This leads to the following definition:

Definition 3. Let $(\mathcal{M}, E, \mathcal{U})$ be a normal strategic-form game. We say that the strategy profile $\mathbf{p}$ is an *equilibrium* if, $\forall m \in \mathcal{M}$, and any $\mathbf{q}_m \in E_m$

$$u_m(\mathbf{p}_m, \mathbf{p}_{m-}) \geq u_m(\mathbf{q}_m, \mathbf{p}_{m-}) \tag{2}$$

Equilibria are defined in terms of comparisons between two points in the action space E which differ only in the same component. If, for any particular point in E, pairwise comparisons can be made with all the points which differ only in one component to the effect that the utility derived from these other points is lower than for the original point, then the particular point is an equilibrium. This fact will be helpful when we decide how to construct the flow graph in section 3.

We distinguish between *pure* and *mixed* equilibriums. In a *pure-strategy equilibrium* players may choose with probability 1 one of their available actions in order to achieve a most favourable outcome. In a *mixed-strategy equilibrium* players must randomise over their available actions to maximise their chances of a favourable outcome. We can make more rigorous definitions in the following manner:

Definition 4. Let $\mathbf{p}$ be a fixed strategy profile in E. If $\mathbf{p}$ fulfills the conditions in Def. 3, then $\mathbf{p}$ is a *pure-strategy equilibrium.*

Definition 5. Let $\pi(\mathbf{p})$ be a probability distribution over all $\mathbf{p} \in E$ (that is, a vector describing the probability that each player will pick exactly the actions described by strategy profile $\mathbf{p}$). Let

$$\mathcal{P} = \{\pi : \sum_{\mathbf{p} \in E} \pi(\mathbf{p}) = 1,\ \pi(\mathbf{p}) \geq 0\}$$

be the family of all such probability distributions over E, and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space where $\mathbf{p}$ and $\mathbf{p}'$ are random vectors with corresponding distributions π' and π in $\mathcal{P}$. Then if

$$\mathbb{E}(u_m(\mathbf{p})) \geq \mathbb{E}((u_m(\mathbf{p}'))$$

for all $m \in \mathcal{M}$ (cf. Def. 3) we say that π is a *mixed-strategy equilibrium.*

Normally, we work with both mixed-strategy and pure-strategy equilibria. Rather than adjust the notation, we will use the notation for pure-strategy equilibria for both forms of equilibria. It will be indicated in the text when we need to make a distinction. Mathematical investigations into optimal behaviours of players in *two-person*, *zero-sum* games were initiated by von Neumann and Morgenstern in 1944. These games are defined so that $\mathcal{M} := \{a, b\}$, E consists of the choices available for each player in the game, and for $\forall \mathbf{p} = (\mathbf{p}_a, \mathbf{p}_b) \in E$ it is true that $u_a(\mathbf{p}_a, \mathbf{p}_b) = -u_b(\mathbf{p}_a, \mathbf{p}_b)$, where we have used $\mathbf{p}_a$ and $\mathbf{p}_b$ to denote the choices of players a and b codified in $\mathbf{p}$ respectively. Solving these types of games implies that if one of the players wins, the other loses.

A seminal result from American mathematician John Nash in 1953, its formulation taken here from [5] with a proof from [8], confirms that all games have at least one mixed-strategy equilibrium:

Theorem 1 (Nash's theorem)**.** Every finite normal strategic-form game has a mixed-strategy equilibrium.

Sketch of proof. Nash's theorem relies on fixed-point theorems developed by Brouwer and/or Kakutani (either can be used, although a proof depending on Brouwer is slightly more involved). Assume the conditions of Def. 4 and 5 are satisfied.

The action space E can be shown to be non-empty, finite-dimensional, compact and convex. We can then additionally find a closed, set-valued mapping $\psi : E \mapsto \mathcal{P}$, such that the image set of this mapping is a non-empty, convex subset of $\mathcal{P}$. Now Kakutani's theorem implies that there is a fixed point $\psi(\mathbf{p}) = \mathbf{p}$. This fixed point is a mixed-strategy equilibrium. □

To help the presentation below we include the following definitions:

Definition 6. For any $m \in \mathcal{M}$, two strategy profiles $\mathbf{p}, \mathbf{q} \in E$ are *m-comparable* if they differ only in the m^{th} component.

Definition 7. If two strategy profiles $\mathbf{p}, \mathbf{q} \in E$ are m-comparable, their *pairwise comparison* is defined to be the difference $|u_m(\mathbf{p}) - u_m(\mathbf{q})|$ for $u_m \in \mathcal{U}$.

We will later use X to denote a pairwise comparison.

Now we can define a potential game. We will search for spaces containing such games because of the important qualities they possess and which we will immediately clarify. The definition and theorem are taken from [3, Ch. 6.4.2] but presented in a terminology consistent with the rest of the present work.

Definition 8. A game $(\mathcal{M}, E, U)$ is a *potential game* if there exists a function $\phi : E \mapsto \mathbb{R}$ such that $\forall m \in \mathcal{M}$ and $\forall \mathbf{p}, \mathbf{p}' \in E$ which are m-comparable,

$$u_m(\mathbf{p}) - u_m(\mathbf{p}') = \phi(\mathbf{p}) - \phi(\mathbf{p}'). \tag{3}$$

Theorem 2. Every potential game which is equipped with a finite action set E, has a pure-strategy Nash equilibrium.

Proof. Def. 8 implies that the utility vectors $u(\mathbf{p}) = (u_1(\mathbf{p}), ..., u_n(\mathbf{p})) \in \mathbb{R}^n$ can be ordered. A pure-strategy equilibrium of a potential game will correspond to the global maximum of the potential function. Indeed, suppose $\mathbf{p}'$ corresponds to the global maximum of the game. Then, for any $m \in \mathcal{M}$ we must have that $\phi(\mathbf{p}') - \phi(\mathbf{p}) \geq 0$. By definition of a potential game, we therefore have $u_m(\mathbf{p}') - u_m(\mathbf{p}) \geq 0$. This confirms a pure-strategy equilibrium in the point $\mathbf{p}'$. □

By Theorem 2 every finite potential game is known to have a pure-strategy Nash equilibrium. In cases where we have a near-potential or difficult-to-solve game, decomposing the game may help us find a potential game, which we then know will give us clear indications of how to optimise our play.

3 Games and flow graphs

Games are sometimes represented in graph form to make them more tangible. In the *extensive-form representation* of a repeated game or a multi-stage game, we may use tree graphs called *Kuhn trees* to represent the how different choices made by players lead to different outcomes. In these cases, each of instances where choices are made are represented as nodes, and the outcomes are represented as terminal nodes. Definitions and examples may be found in [4, Sec. 5.3] and [3, Fig. 5.1-5.2]. In [9], graphs have additionally been used to indicate relationships between players, such as one player being able to punish a different player from deviating from the globally good action (namely, the action which causes the most desirable spread of utility between players, rather than the action which maximises an individual player's utility).

The purpose of the graph representation in this text is to locate sets of pure and mixed equilibria in sets of games. The core of the theory in this text is taken from [2].

Since normal strategic-form games are solved exactly by looking at pairwise comparisons (recall 3), we may represent the pair of strategic choices to be compared as two vertices connected by an edge. Each vertex denotes a specific choice of strategies by each of the players. A pairwise comparison is defined to be possible if only one of the components in two vertices is different from the other. If two vertices have the quality that between them only one component is changed, we say there is an edge between those vertices.

The change in utility which occurs when a player chooses between different actions is represented as a flow on the edge. We will construe these flows as a field over the graph. Cycles in the flow are mixed-strategy equilibria - situations in which for every choice of action in the cycle, at least one player will benefit from picking a different action. Sinks in the flow are pure-strategy equilibria - no player can benefit from picking a different action from the one which they have picked in the sink vertex.

Using relationships between graph theory, linear algebra och vector calculus, we can then use Helmholtz decomposition to analyse the game structures more closely by separating out different kinds of flows that emerge in the game graph.

We will also present the following theorem, taken from [2, Thm 3.1], which is used to decompose a space of games into potential (pure-strategy solveable) and harmonic (mixed-strategy solveable) parts::

Theorem 3 (Helmholtz Decomposition)**.** Let C_0, C_1 and C_2 be three vector spaces equipped with inner products, related in such a way that $\delta_0 : C_0 \mapsto C_1$ and $\delta_1 : C_1 \mapsto C_2$ are linear operators with adjoints δ_0^* and δ_1^*. Then the vector space C_1 admits an orthogonal decomposition

$$C_1 = \mathrm{Im}(\delta_0) \oplus (\ker(\delta_1) \cap \ker(\delta_0^*)) \oplus \mathrm{Im}(\delta_1^*) \tag{4}$$

Helmholtz' decomposition is a tool from vector calculus commonly used in the physical sciences to study fields. It implies that a vector field on a bounded domain can be decomposed into a curl-free and a divergence-free part. Since we will later define C_1 to be the space of pairwise comparisons, and C_1 is bounded due to the games being finite, we can (and shall) decompose C_1 using the Helmholtz theorem.

4 Application

4.1 Currency exchange markets

Preference rankings can be described in the terminology of game theory. Specifically, a preference ranking game is a game in which we determine an ordering over the of players with respect to how preferred each player is to the other.

In [1] an exemplification of Hodge decomposition is given for ranking of a currency exchange market, according to a model for such a market developed by Ma in [14]. We can understand the original theory to be interesting for the application of real-time monitoring of foreign currency exchanges in order to find exploitable arbitrage opportunities (the opportunity to "get something for nothing", see [15]). In this simpler version of the problem, we make use of the assumption that there is an arbitrage-free baseline, a gold-standard, for the currencies, which we can use to create an objective universal ranking for the currencies. In our example, the purpose is to rank five European currencies internally (from most to least "preferred"), using data from `xe.com`. The exchange rates collected from `xe.com` are found in Table 1. The theoretical framework is explained in the terminology used by [2].

This is a five-player game in which five currencies,

$$\mathcal{M} = \{\mathrm{EUR}, \mathrm{GBP}, \mathrm{SEK}, \mathrm{RON}, \mathrm{BGN}\}$$

may be exchanged for one another. The relationship of $m \in \mathcal{M}$ being preferred to $k \in \mathcal{M}$ is expressed as $m \succeq k$. Equivalently if $m \in \mathcal{M}$ is not preferred to $k \in \mathcal{M}$ we write $m \preceq k$. A preference ordering on the set $\mathcal{M}$ means that $m_{(0)} \succeq m_{(1)} \succeq m_{(3)} \succeq m_{(4)} \succeq m_{(5)}$ for some permutation of elements $m_i \in \mathcal{M}$.

Index the set $\{\mathrm{EUR}, \mathrm{GBP}, \mathrm{SEK}, \mathrm{RON}, \mathrm{BGN}\}$ by $\{1, 2, 3, 4, 5\}$. For each comparison between currencies the following holds: one Bulgarian lev is worth €0.51 or £0.39 by Table 1. Using the indices $2, 1$ and 5 for euro, pounds and lev respectively, we assume that we can find constants s_5, s_2 and s_1 such that €$0.51 = \frac{s_1}{s_5}$ and £$0.39 = \frac{s_2}{s_5}$. This gives us a vector

$$p_5 = \left(\frac{s_1}{s_5}, \frac{s_2}{s_5}, \frac{s_3}{s_5}, \frac{s_4}{s_5}, \frac{s_5}{s_5}\right)$$

	€	£	SEK	RON	BGN
€	1	0.77	9.30	4.47	1.95
£	1.29	1	11.99	5.76	2.52
SEK	0.10	0.08	1	0.48	0.21
RON	0.22	0.17	2.08	1	0.44
BGN	0.51	0.39	4.75	2.28	1

Table 1: The amount of row currency which is acquired from one unit of a column currency.

	€	£	S	R	B
€	0.0	0.316	-2.81	-1.893	-0.861
£	-0.316	0.0	-3.126	-2.21	-1.178
S	2.81	3.126	0.0	0.917	1.949
R	1.893	2.21	-0.917	0.0	1.032
B	0.861	1.178	-1.949	-1.032	0.0

Table 2: The geometric mean matrix of the currency exchange described in Table 1. It is symmetric, so $(\delta_1 M)$ evaluates to 0.

for the lev. Constructing for each of the five currencies such a vector, the action space E may be represented in matrix form as $[\ p_1\ ;\ p_2\ ;\ p_3\ ;\ p_4\ ;\ p_5\ ]$, with each p_i being a row. The numerical values of E represented in this way are listed in Table 1. The constants s_i are invariant, but may not be unique. The important thing for our ranking is their fixed relationship to one another.

To solve for a universal preference ranking, for each of the elements in Table 1 we apply a logaritmic transform:

$$\log\left(\frac{s_i}{s_k}\right) = \log(s_j) - \log(s_k), \quad i, j, k \in [1, 2, 3, 4, 5] \tag{5}$$

The result is given in Table 2, and it represents the set of (evaluated) utility functions $u_m \in \mathcal{U}$.

Now recall Def. 6. Let W^m be an indicator function, indicating whether two strategies are m-comparable. $W^m(\mathbf{p}, \mathbf{q}) = 1$ exactly when, if $\mathbf{p} = \frac{s_i}{s_j}$, then $\mathbf{q} = \frac{s_i}{s_k}$ or $\mathbf{q} = \frac{s_k}{s_j}$.

Denote by C_0 the space of all real-valued functions on E. The utility functions u_m will be contained in C_0, and a generic member of C_0 we will denote by φ. On the space C_0, we define a *combinatorial gradient operator* δ_0 (see [2, p. 9]) as

$$(\delta_0 \varphi_m) = \begin{cases} W^m(\mathbf{p},\mathbf{q})(\varphi_m(\mathbf{p}) - \varphi_m(\mathbf{q})) & \mathbf{p},\mathbf{q} \in E \\ 0 & \text{else.} \end{cases} \tag{6}$$

We can see how the generalisation $(\delta_0 \varphi) = W(\mathbf{p},\mathbf{q})(\varphi(\mathbf{p}) - \varphi(\mathbf{q}))$ is possible and motivated for any currency $m \in \mathcal{M}$.

We will denote by C_1 the space of possible pairwise comparisons of members of C_0, so that elements of the space C_1 can be written as the functions $X(\mathbf{p},\mathbf{q}) = W(\mathbf{p},\mathbf{q})(\varphi(\mathbf{p}) - \varphi(\mathbf{q}))$. We define on this space a *curl operator* δ_1 (see [2, p. 9]) such that

$$(\delta_1 X) = \begin{cases} X(\mathbf{p}_1,\mathbf{p}_2) + X(\mathbf{p}_2,\mathbf{p}_3) + X(\mathbf{p}_3,\mathbf{p}_1) & \text{if } W(\mathbf{p}_i,\mathbf{p}_j) = 1, \\ 0 & \text{else.} \end{cases} \tag{7}$$

Referring back to Def. 8, we observe that for φ such that $\delta_0 \varphi = X$, we may consider φ the potential function of X, and we also say that X is globally consistent.

Denote by C_2, similarly, the space of all possible comparisons of two currency exchanges, for instance that we start with one euro, convert it to Romanian lei and then convert the lei into British pounds sterling. The members of the space C_2 are functions $T(\mathbf{p},\mathbf{q},\mathbf{r}) = X(\mathbf{p},\mathbf{q}) + X(\mathbf{q},\mathbf{r}) + X(\mathbf{r},\mathbf{p})$. It is clear that members of the space C_2 coincide with the results of applying the curl operator onto elements of the space C_1. If a T-term in C_2 evaluates to 0, we say that X is locally consistent.

Remark 1. In a pairwise comparison, we need only to give attention to the interaction of any two currencies at once. A strategy profiles $\mathbf{p}$ will describe one currency in relation to one other currency. Let m index the currency evaluated with respect to the currency indexed n. If the utility function $u_m(\mathbf{p})$ evaluated to $(\log s_n - \log s_m)$, it must be true that $T(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = (\log s_j - \log s_i) + (\log s_i - \log s_k) + (\log s_k - \log s_j) = 0$ for any three distinct $\mathbf{p}_i \in E$. It is therefore true that X is locally consistent for every point in E. It follows that X is globally consistent.

It is verifiable that C_0, C_1 and C_2 are vector spaces: normal scalar addition and multiplication rules apply. In addition, the spaces can be equipped with inner products. Both [1] and [2] construct inner products with respect to an l_2-norm, and [1] additionally provides a construction of inner products with respect to an l_1-norm.

From the global and local consistency, we can derive that the image set of the gradient operator, $\mathrm{Im}(\delta_0)$, is equal to the kernel of the curl operator, $\ker(\delta_1)$, which in turn is equal to all of C_1. By theorem 3 we see that C_1 decomposes into itself and two empty sets.

An adjoint of a linear operator is unique with respect to the inner product, and by theorem 3 we have that the adjoint δ_0^* of δ_0 has the property

$$\begin{aligned}(\delta_0^* X)(\mathbf{p}) &= -\sum_{\mathbf{q}\in E} W(\mathbf{p},\mathbf{q})(\varphi(\mathbf{p}) - \varphi(\mathbf{q})) \\ &= -\sum_{\mathbf{p}^m \in E} u_m(\mathbf{p}^m)\end{aligned}$$

We understand the expression on the right hand side to be the sum of all values of the utility function for currency m given its degree of preference to other currencies. The operator $-\delta_0^*$ is a *divergence operator*. The remainder of the problem of applying the algorithm devised in [1, Theorem 5.1] (implementation in A.2.1) involves solving the equation

$$s^* = -\Delta_0^{\dagger}(\delta_0^* X) \tag{8}$$

where $\Delta_0^{\dagger}$ is the Moore-Penrose pseudo-inverse (see Appendix A.1.1) of the Laplacian operator defined as

$$\Delta_0 = \delta_0^* \circ \delta_0. \tag{9}$$

This definition implies that

$$[\Delta_0]_{\mathbf{p},\mathbf{q}} = \begin{cases} \sum_{\mathbf{r}\in E} W(\mathbf{p},\mathbf{r}) & \text{if } \mathbf{p} = \mathbf{q} \\ -1 & \text{if } \mathbf{p} \neq \mathbf{q}\text{ , } \mathbf{p},\mathbf{q} \text{ pairwise comparable.} \\ 0 & \text{else.} \end{cases} \tag{10}$$

Following the above procedure, we find that the global ranking based on geometric mean is $(1.0496, 1.366, -1.7604, -0.8436, 0.1884)$. This rankings indicates that the order of preference pounds $\succeq$ euro $\succeq$ lev $\succeq$ leu $\succeq$ krona, which conforms to our expectation.

4.2 Decompositions of congestion games

Using the construction of a space of flows over a graph in relation to a game, as described in the previous example, is already developed in [2] into a complete decomposition of a game for two players. We recall that a normal strategic-form game may be presented in matrix form. Such a matrix is called a *bimatrix*, since the sets of utilities for each of the players in themselves can be placed in a matrix (see in particular Table 3 and Tables 4 and 5). Our goal in the following section is to solve a general congestion game for two players and three resources with the method given.

First we require some more theory. The following definition of a congestion game is borrowed from [7]. The construction of the normal strategic form follows the description in [16, Sec. 4.1.3].

Definition 9 (Congestion game)**.** A *general weighted congestion game* is defined by a set of players $\mathcal{M}$, a set of resources $\mathcal{R}$, and a set of non-negative, non-decreasing cost functions $\mathcal{C} = \{c_r : \mathbb{R}_+ \mapsto \mathbb{R}_+\}$. It is the tuple $(\mathcal{M}, \mathcal{R}, \mathcal{C})$.

For each player $m \in \mathcal{M}$ we specify a weight w_i and an action set E_m, where w_i is typically in $\mathbb{R}_+$. The *congestion* on a resource is the total sum of weights put on that resource when players use it. An *outcome* is a set of actions $\mathbf{p} = (p_1, ..., p_M) \in E$. The congestion on a resource $r \in \mathcal{R}$ following the action $\mathbf{p}$ is given by

$$g_r(\mathbf{p}) = \sum_{m \in \mathcal{M}} w_m, \quad \mathbf{p} \in E, \tag{11}$$

The *cost* function associated with a player $m \in \mathcal{M}$ is the solution to the equation $c_m(\mathbf{p}) = w_m \sum_{r \in p_m} c_r(g_r(\mathbf{p}))$.

We will consider a congestion game for two players and three resources. Either both of the players choose the same resource at the same time, or they choose different resources. We have set $w_m = 1$ to simplify notation. Note that *resources* is a contextual adaptation of the term *actions* (i.e. the action to use a certain resource) and that the *cost* function of a player may be interpreted as *utility*. While normally we would seek to maximise utility, in a congestion game we seek to minimise cost. These relations imply that we can represent the game in normal strategic form. Such a representation can be found in Table 3, and it is decomposed into two matrices in Tables 4 and 5.

The operators δ_0 and δ_1 defined in Eq. 6 and 7 respectively are not restricted to a specific action space of a game. We will introduce some operators which are restricted to actual game we have defined.

Let $\mathcal{G}$ be a game tuple with a fixed set of players $\mathcal{M}$, who may, be choosing different actions from the set E end up with pre-determined utilities $u \in \mathcal{U}$. With the notation from Sec. 4.1, let $X = \sum_{m \in \mathcal{M}} D_m u_m$ be the

Table 3: (Bimatrix) Normal strategic form representation of a congestion game for two players and three resources.

	r_1	r_2	r_3
r_1	$((c_1+c_2), (c_1+c_2))$	(c_1, c_2)	(c_1, c_2)
r_2	(c_1, c_2)	$((c_1+c_2), (c_1+c_2))$	(c_1, c_2)
r_3	(c_1, c_2)	(c_1, c_2)	$((c_1+c_2), (c_1+c_2))$

pairwise comparison (recall Def. 7). Letting $D = [D_1, ..., D_n]$, where $n = |\mathcal{M}|$ (the finite cardinality of the set of players), we have an operator $D : C_0' \mapsto C_1$, where C_0' is the space of actual utility functions. Since $u \in C_0'$, we can write $\sum_{m\in\mathcal{M}} D_m u_m = Du$.

The operators δ_0 and D are related by a scaling factor Λ such that $D = \Lambda\delta_0$ while $D^* = \delta_0^*\Lambda$. Also we need [2, Theorem 4.1], which is the main theorem used for the decomposition of games:

Theorem 4 (Game decomposition)**.** Let $\mathcal{G}_{\mathcal{M},E}$ be the space of games having the set of players $\mathcal{M}$ and the space of actions E. This space is a direct sum of its potential, harmonic and non-strategic subspaces, namely

$$\mathcal{G}_{\mathcal{M},E} = \mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N} \tag{12}$$

In particular, if a game $G \in \mathcal{G}_{\mathcal{M},E}$ has utilities $U = \{u_m\}, m \in \mathcal{M}$, the elements of these subspaces are of the form

$$U_P = D^\dagger \delta_0 \delta_0^\dagger D u \tag{13}$$

$$U_H = D^\dagger (I - \delta_0 \delta_0^\dagger) D u \tag{14}$$

$$U_N = (I - D^\dagger D) u \tag{15}$$

where $U_P \in \mathcal{P}$, $U_H \in \mathcal{H}$ and $U_N \in \mathcal{N}$, and additionally $U_P + U_H + U_N = U$.

It is the proof of this theorem which relies on Theorem 3 stated above.

Table 4: Player 1 matrix A

	r_1	r_2	r_3
r_1	c_1+c_2	c_1	c_1
r_2	c_1	c_1+c_2	c_1
r_3	c_1	c_1	c_1+c_2

Table 5: Player 2 matrix B

	r_1	r_2	r_3
r_1	c_1+c_2	c_2	c_2
r_2	c_2	c_1+c_2	c_2
r_3	c_2	c_2	c_1+c_2

From [2] we know that the bimatrix (A, B) can be decomposed into potential and harmonic parts, (A_P, B_P) and (A_H, B_H) respectively. We also know from that the result in Theorem 4 provides that

$$(A_P, B_P) = (S + \Gamma, S - \Gamma), \qquad (A_H, B_H) = (D - \Gamma, -D + \Gamma) \tag{16}$$

where $S = \frac{1}{2}(A + B)$, $D = \frac{1}{2}(A - B)$ and $\Gamma = \frac{1}{2h}(A\mathbf{1}(\mathbf{1}^T) - \mathbf{1}(\mathbf{1}^T)B)$, where h is the equivalent number of available actions to each player (in this case 3), and $\mathbf{1}$ is the h-dimensional vector of ones. We can compute

$$D = \frac{1}{2}(c_1 - c_2)\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \Gamma, \qquad S = \frac{1}{2}(c_1 + c_2)\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{17}$$

and find that the harmonic component is the zero matrix, while the potential component is in fact the original normal strategic-form matrix from Table 3. This demonstrates a well-known result that every congestion game is a potential game (see [3, Ch. 6]).

4.3 Decomposition of a game for three players

Our final endeavour is to decompose a (normalised) game with

$\mathcal{M} = \{a, b, c\}$,

$E = \{a_1, a_2, a_3\} \times \{b_1, b_2\} \times \{c_1, c_2\} = E_a \times E_b \times E_c$ and

$\mathcal{U} = \{u_m : E \mapsto \mathbb{R}\ , m \in \mathcal{M}\}$.

Namely, a three-player game, where the first two players have two actions each to choose from, but the third player has three options to choose from. The action space E is shown in Fig. 1. There is no non-strategic information in this game, in the meaning of "normalised games" given in [2, Def. 4.1]. This means that $D^\dagger D U = U$.

We now turn to the application of Theorem 4. First, we may observe that the utility vectors $U = (u_a, u_b, u_c), u_m \in \mathcal{U}$ (Fig. 2) at each node $\mathbf{p} \in E$ (Fig. 2) may be represented in the form of a tri-matrix $U = (A, B, C)$, where A is a matrix containing all the real-valued outputs of evaluating the function u_a at e_{abc}, with similar reasoning for matrices B and C.

Recalling Eq. (9) and using the fact that D^* and D by the definition of adjoints have orthogonal image spaces, we see that

$$\Delta_0 = \delta_0^* \delta_0 = \sum_{m \in \mathcal{M}} D_m^* D_m \tag{18}$$

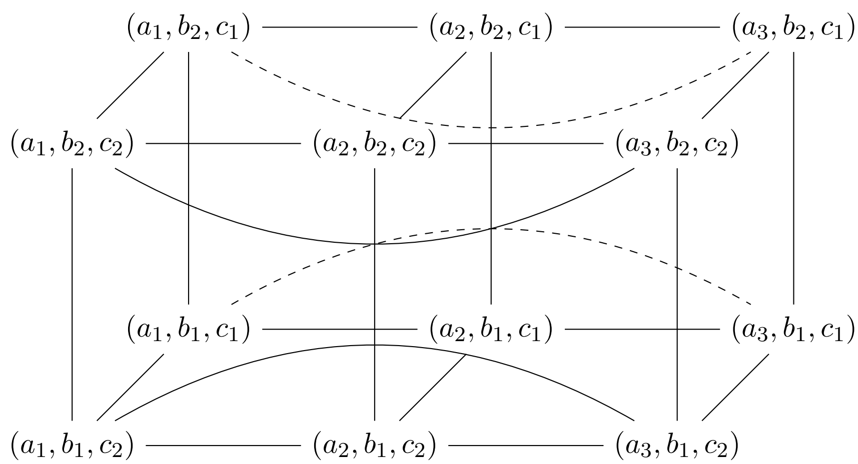


Figure 1: This is the strategy space E of the game we are studying in this section. The edges represent points in E which are m-comparable with respect to functions $\varphi \in C_0$.

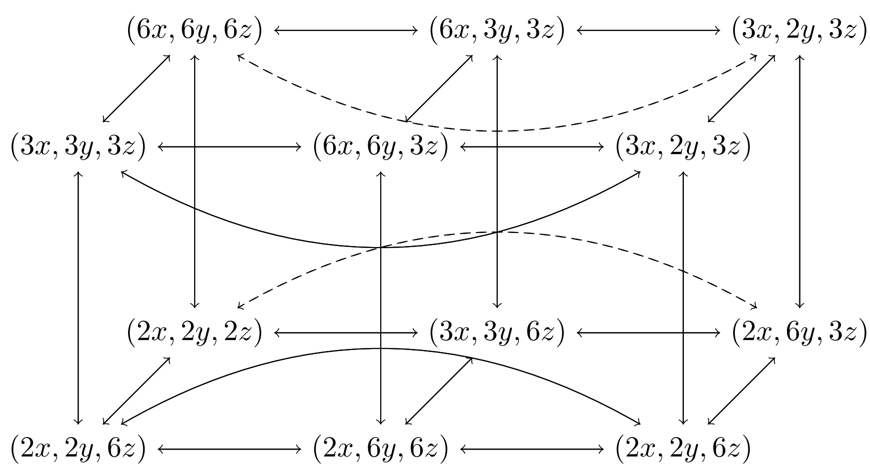


Figure 2: This is a graph of "utility flows" as players switch between actions in E. Each point from Fig. 1 has been mapped to a triplet of functions $U = (u_a, u_b, u_c)$ where $u_i \in \mathcal{U}$. For the purpose of our investigation, the u_i have been chosen arbitrarily by the author, and later we will put $(x, y, z) = (3, 2, 2)$ to find potential and harmonic components of the game.

	1	2	3	4	5	6	7	8	9	10	11	12
1	4	-1	-1	0	-1	0	0	0	-1	0	0	0
2	-1	4	0	-1	0	-1	0	0	0	-1	0	0
3	-1	0	4	-1	0	0	-1	0	0	0	-1	0
4	0	-1	-1	4	0	0	0	-1	0	0	0	-1
5	-1	0	0	0	4	-1	-1	0	-1	0	0	0
6	0	-1	0	0	-1	4	0	-1	0	-1	0	0
7	0	0	-1	0	-1	0	4	-1	0	0	-1	0
8	0	0	0	-1	0	-1	-1	4	0	0	0	-1
9	-1	0	0	0	-1	0	0	0	4	-1	-1	0
10	0	-1	0	0	0	-1	0	0	-1	4	0	-1
11	0	0	-1	0	0	0	-1	0	-1	0	4	-1
12	0	0	0	-1	0	0	0	-1	0	-1	-1	4

Table 6: This is $\Delta_0 = \Delta_{0,a} + \Delta_{0,b} + \Delta_{0,a}$ colour-coded in such a way that $\Delta_{0,a}$ is red, $\Delta_{0,b}$ is blue and $\Delta_{0,c}$ is magenta. The numbering follows the order in Fig. 3b. Each black 4 is equal to 2 + 1 + 1.

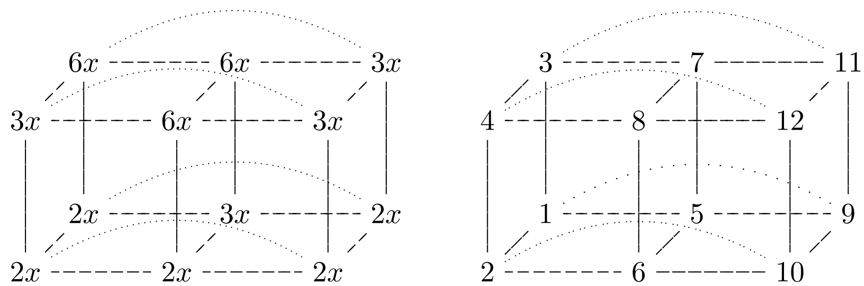
We will also recall the following properties of these operators and their adjoints from [2, Lemma 4.4]:

Theorem 5. Let h_m be the cardinality of E_m, or equivalently, the number of actions available for player m and let M be the number of players. The pseudo-inverses of operators D_m and D satisfy, in addition to the requirements listed in Appendix A.1.1, the following conditions

$$i)\ D_m^\dagger = \frac{1}{h_m} D_m^*,\ ii)\ D^\dagger = \begin{pmatrix} D_1^\dagger \\ \vdots \\ D_M^\dagger \end{pmatrix},\ iii)\ D^\dagger \delta_0 = D^\dagger D,\ iv)\ \delta_0^\dagger D U = \Delta_0^\dagger(\Delta_0 \cdot U)$$

It immediately follows from the rules of our game defined above that $M = 3$, $h_1 = 3$, $h_2 = 2$ and $h_1 = 2$. We will also introduce a decomposition of the Laplacian operator Δ_0, such that $\Delta_{0,m}$ contains exactly the information about m-comparable strategies for player m. In Fig. 3 we see the A from the (A, B, C)-triplet, as well as the order in which strategies are considered for the construction of the laplacian in Table 6. We will use this order to make vectors of length 12 out of the strategies in A, B, C and apply the operators to these vectors.

Now we are ready to apply to results from Theorem 4 directly. First, we find an expression for the potential component, which we then compute.



(a) A (b) Ordering of strategies.

Figure 3: On the left are the utilities of player a from Fig. 2. We also show the ordering of the strategies for the Laplacian Δ_0 for Table 6. By ordering thusly the nodes of A above, we with compose Δ_0 with a resulting vector A.

$$
\begin{aligned}
U_P &= D^\dagger \delta_0 \delta_0^\dagger D U = (D^\dagger D)\Delta_0^\dagger(\Delta_{0,a}A + \Delta_{0,b}B + \Delta_{0,c}C) \\
&= \left(\frac{D_1^* D_1}{h_1}, \frac{D_2^* D_2}{h_2}, \frac{D_3^* D_3}{h_3}\right)\Delta_0^\dagger(\Delta_{0,a}A + \Delta_{0,b}B + \Delta_{0,c}C) \\
&= \left(\frac{\Delta_{0,a}}{3}, \frac{\Delta_{0,b}}{2}, \frac{\Delta_{0,c}}{2}\right)(\Delta_0^\dagger\Delta_{0,a}A + \Delta_0^\dagger\Delta_{0,b}B + \Delta_0^\dagger\Delta_{0,c}C)
\end{aligned}
$$

where we have used property $i)$ of Theorem 5 and the observation from Eq. (18). Now we turn to the harmonic component.

$$
\begin{aligned}
U_H &= D^\dagger(I - \delta_0\delta_0^\dagger)DU \\
&= D^\dagger I D U - D^\dagger \delta_0 \delta_0^\dagger D U \\
&= D^\dagger D U - D^\dagger \delta_0 \delta_0^\dagger D U \\
&= U - D^\dagger \delta_0 \delta_0^\dagger D U \\
&= (A, B, C) - U_P
\end{aligned}
$$

where we have again used property $i)$ of Theorem 5.

It follows that if $D^\dagger D U = U$, then $U_N = (I - D^\dagger D)U = 0$. It is quite clear that the direct sum property $U_P \oplus U_H \oplus U_N = U = (A, B, C)$ holds.

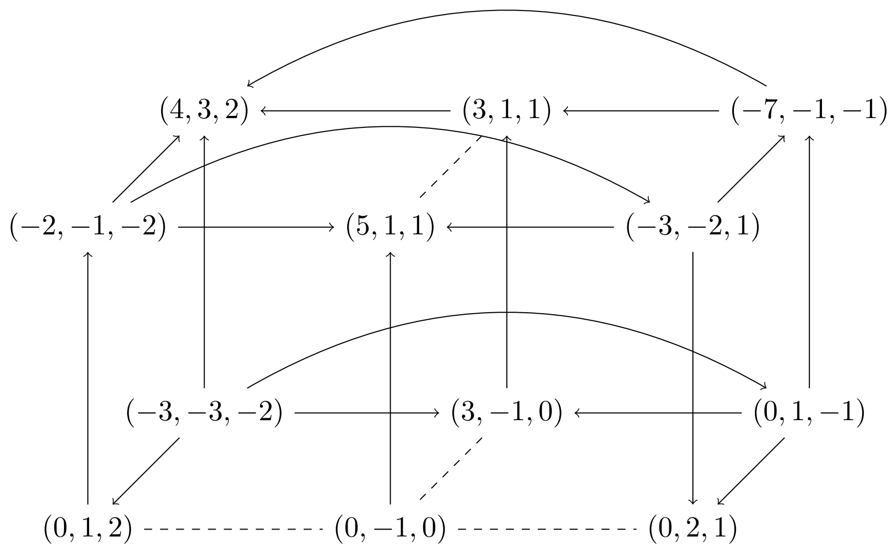


Figure 4: The potential part U_P of the utility flows in Fig. 2 for $(x, y, z) = (3, 2, 2)$. $(4, 3, 2)$ in the upper, back left corner is a sink, and therefore a pure strategy equilibrium. Dashed lines mean indifference.

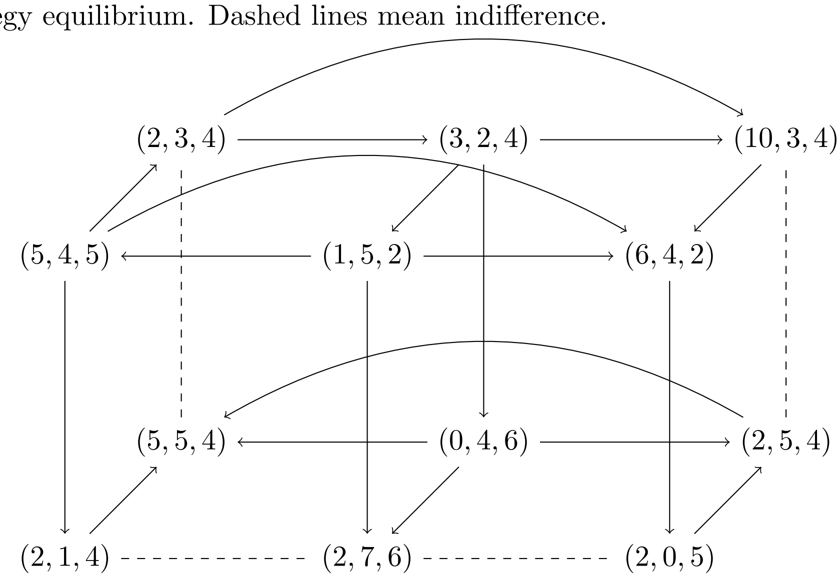


Figure 5: The harmonic part U_H of the utility flows in Fig. 2. Dashed lines mean indifference. There is no definitive sink in this graph. From every node there is at least one flow going outward, or a path of indifference.

5 Discussion

Using matrix theory to investigate decompositions of games may be useful in some circumstances, but the lack of efficient algorithms for matrix calculations reduces the traction of the model. So, for instance, while congestion games present many real-life challenges in network or traffic management, this model is unlikely to be helpful in resolving those challenges.

In the decomposition of a game with three players, each having between two and three actions to choose from, the Laplacian was already quite large (12×12). It will grow rapidly as the action sets of the players grow. While the Laplacian as such is sparse, it's pseudo-inverse is not. The computational complexity thus remains high.

The currency ranking was used here to get a feel for the model. In practical terms, we may be more interested in locating actual arbitrage opportunities rather than finding an objective, neutral ranking of the currencies under the assumption of an outside observer.

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A Appendix

A.1 Further definitions

A.1.1 Moore-Penrose pseudo-inverse

Definition 10 (Moore-Penrose Pseudo-inverse)**.** A pseudo-inverse of a matrix A is a matrix $A^\dagger$ such that $AA^\dagger A = A$ without $AA^\dagger$ necessarily being the identity matrix I. To qualify as a Moore-Penrose pseudo-inverse, this matrix has to fulfill the following four conditions:

i) $AA^\dagger A = A$

ii) $A^\dagger AA^\dagger = A^\dagger$

iii) $(AA^\dagger)^* = AA^\dagger$

iv) $(A^\dagger A)^* = A^\dagger A$

The product of a matrix with its Moore-Penrose pseudo-inverse is its own adjoint, and the product of the pseudo-inverse with its matrix is as well. This definition is taken from Wikipedia, since it was not given in any of the literature which I consulted.

A.2 Python code

A.2.1 Currency rankings

This is the code I used to produce the arithmetic and geometric mean matrices for the currency exchanges. The extracted exchange rates are hardcoded into the code, as is the laplacian operator.

```
from numpy import *
from scipy import *


# Inserting the currency matrix
M1 = array([[1.0, 0.77, 9.30, 4.47, 1.95],
                        [1.29, 1.0, 11.99, 5.76, 2.52],
[0.10, 0.08, 1.0, 0.48, 0.21],
[0.22, 0.17, 2.08, 1.0, 0.44],
[0.51, 0.39, 4.75, 2.28, 1.0]])
```

```
W_geom = []
# Constructing the geometric mean matrix
for j in range(0,len(M1[:,1])):
    V = []
    # per currency pairwise comparisons vector
    for i in range (0,len(M1[:,1])):
        Y = (sum(log(M1[:,j])-log(M1[:,i])))/(len(M1[:,1])-1)
        V.append(Y)
        # append each pairwise comparison to the vector
    W_geom.append(V)
    # append each vector to the matrix.
W_geom_round = around(W_geom, 3)
# print("The geometric mean is \n {W}".format(W=W_geom_round))

#Divergence of geometric mean
Y = 0
for j in range(0,len(W_geom_round[j,:])):
    Y = Y + W_geom_round[j,:]
# print(Y)

# Laplacian matrix
D0 = array([[4.0, -1.0, -1.0, -1.0, -1.0],
            [-1.0, 4.0, -1.0, -1.0, -1.0],
[-1.0, -1.0, 4.0, -1.0, -1.0],
[-1.0, -1.0, -1.0, 4.0, -1.0],
[-1.0, -1.0, -1.0, -1.0, 4.0]])

print("Global ranking is {geom} for geometric mean".
                    format(geom=dot(linalg.pinv(D0),Y)))
```